

Week 9

Theorem: let $1 \leq p < +\infty$, $x_0 \in X$. let $\{\mu_n\}_n \in \mathcal{P}_p(X)$, $\mu \in \mathcal{P}_p(X)$.

Then, it is equivalent:

① $\mu_n \rightarrow \mu$ narrowly and $\int_X d^p(x, x_0) d\mu_n \rightarrow \int_X d^p(x, x_0) d\mu$

② $W_p(\mu_n, \mu) \rightarrow 0$

Remark: let $\{\mu_n\}_n \in \mathcal{P}(X)$, $\mu \in \mathcal{P}(X)$. Then: $\mu_n \xrightarrow{*} \mu$ weakly* $\Leftrightarrow \mu_n \rightarrow \mu$ narrowly.

Proof: we showed ② $\Rightarrow$ ① in general, ① $\Rightarrow$ ② for X compact.

① $\Rightarrow$ ② in general. Fix $\delta > 0$, define $M_n := \int (1 + d^p(x, x_0)) d\mu_n$
 $\downarrow$ by assumption
 $M := \int (1 + d^p(x, x_0)) d\mu$

Step 1: $\frac{1 + d(x, x_0)^p}{M_n} \mu_n \rightarrow \frac{1 + d(x, x_0)^p}{M} \mu$ narrowly (*).
 $\cap$
 $\mathcal{P}(X)$.

Indeed, testing with $\varphi \in C_c(X)$, we observe $(1 + d(x_0, x)^p) \varphi(x) \in C_c(X)$.

Use this test in $\mu_n \rightarrow \mu$: $\frac{1}{M_n} \int (1 + d^p(x, x_0)) \varphi d\mu_n \rightarrow \frac{1}{M} \int (1 + d^p(x, x_0)) \varphi d\mu$.

So (*) is true with weak-* convergence. By Remark, also for narrow convergence.

Step 2: By Prokhorov's theorem applied to (*), $\exists K \subseteq X$ compact such that:

$$\int_{X \setminus K} (1 + d^p(x, x_0)) d\left(\frac{\mu_n}{M_n} + \frac{\mu}{M}\right) \leq \delta \quad \forall n \Rightarrow \int_{X \setminus K} (1 + d^p(x, x_0)) d(\mu_n + \mu) \leq 2M\delta \quad \forall n \text{ large}$$

$\hookrightarrow M/M_n \leq 2$

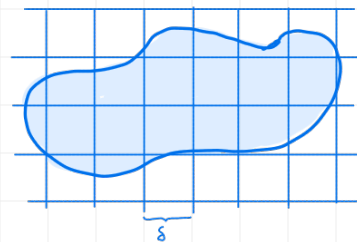
Step 3: let Q_i be a grid of cubes of size δ covering K .

By weak convergence, $\mu_n(Q) \rightarrow \mu(Q)$ (+) for "many" cubes Q (as soon as $\mu(\partial Q) = 0$).

Since X locally compact, we can find a finite family of

$$(\varphi_i)_{i \in I} \in C_c(X), \quad \varphi_i \geq 0, \quad \text{s.t.} \quad \sum_{i \in I} \varphi_i \leq 1 \text{ in } X, \quad \sum_{i \in I} \varphi_i \equiv 1 \text{ in } K,$$

$\text{diam}(\text{supp}(\varphi_i)) \leq \delta$ for all $i \in I$.



Set $\lambda_{n,i} := \int_x \varphi_i d\mu_n$, $\lambda_i := \int_x \varphi_i d\mu$, $\lambda_{n,i} := \min\{\lambda_{n,i}, \lambda_i\}$

and define the measures: $\alpha_{n,i} := \frac{\lambda_{n,i}}{\lambda_{n,i}} \varphi_i d\mu_n$, $\beta_{n,i} := \frac{\lambda_{n,i}}{\lambda_i} \varphi_i d\mu$,

$$\alpha_n := \mu_n - \sum_i \alpha_{n,i}, \quad \beta_n := \mu - \sum_i \beta_{n,i}.$$

Note $\alpha_{n,i}(x) = \beta_{n,i}(x) = \lambda_{n,i}$, $\alpha_n(x) = \beta_n(x) = 1 - \sum_i \lambda_{n,i}$.

Take $\gamma_n := \sum_i \frac{\alpha_{n,i} \otimes \beta_{n,i}}{\lambda_{n,i}} + \frac{\alpha_n \otimes \beta_n}{1 - \sum_i \lambda_{n,i}}$.

We can check:

• $\gamma_n \in \mathcal{P}(\mu_n, \mu)$, $\pi_{1\#} \gamma_n = \sum_i \pi_{1\#} \frac{\alpha_{n,i} \otimes \beta_{n,i}}{\lambda_{n,i}} + \pi_{1\#} \frac{\alpha_n \otimes \beta_n}{1 - \sum_i \lambda_{n,i}}$
 $= \sum_i \alpha_{n,i} + \alpha_n = \mu_n$.

• $W_p(\mu_n, \mu) \leq \int d(x,y)^p d\gamma_n(x,y) =$
 $= \sum_i \int d(x,y)^p \frac{d\alpha_{n,i}(x) d\beta_{n,i}(y)}{\lambda_{n,i}} + \int d(x,y)^p \frac{d\alpha_n(x) d\beta_n(y)}{1 - \sum_i \lambda_{n,i}}$
 $\leq \delta^p \sum_{i \in I} \lambda_{n,i} + 2^p \int_{K \cup K^c} d(x, x_0)^p d\alpha_n(x) + 2^p \int_{K \cup K^c} d(y, x_0)^p d\beta_n(y)$
 $\leq \delta^p + 2^p \sup_K d(x, x_0)^p \alpha_n(K) + \int_{x \in K} d(x, x_0)^p d\alpha_n(x) + \dots$ (Same with β)

$\alpha_n(K) = \mu_n(K) - \sum_i \int_K \frac{\lambda_{n,i}}{\lambda_{n,i}} \varphi_i d\mu_n \xrightarrow{\mu_n \xrightarrow{p} \mu} \mu_n(K) - \int_K \underbrace{\sum_i \varphi_i d\mu_n}_1 = 0$

Same for $\beta_n(x)$.

□

Geodesics in the Wasserstein space

Def: A curve $\eta(t): [0,1] \rightarrow X$ is a (constant speed) geodesic

if $d(\eta(t), \eta(s)) = |t-s| d(\eta(0), \eta(1)) \quad \forall 0 \leq s \leq t \leq 1. \quad (\otimes)$

Remark (motivation): Let M be an (embedded) manifold. A constant speed geodesic between x and $y \in M$ would be

$$\argmin_{\substack{\eta(0)=x \\ \eta(1)=y}} \left(\int_0^1 |\eta'|^2 dt \right)^{1/2} \quad (1) \quad (*)$$

or equivalently: $\argmin_{\substack{\eta(0)=x \\ \eta(1)=y}} \int_0^1 |\eta'| dt \quad (2)$

$\uparrow$ this function is invariant by reparametrization of the curve

(reparametrized to be) constant speed minimizer

$\rightarrow$ namely, $|\eta'| \equiv \text{constant}$

Lemma: $\eta(t)$ geodesic $(*) \Rightarrow (\otimes)$ holds.

$$\begin{aligned} d(\eta(0), \eta(1)) &\leq d(\eta(0), \eta(s)) + d(\eta(s), \eta(t)) + d(\eta(t), \eta(1)) \\ &\leq (s + (t-s) + (1-t)) d(\eta(0), \eta(1)) = d(\eta(0), \eta(1)). \end{aligned}$$

$$\downarrow$$

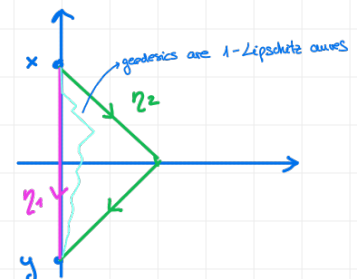
$$|\eta'| \equiv d(\eta(0), \eta(1)), \quad d(\eta(0), \eta(s)) \leq \int_0^s |\eta'|$$

$\rightarrow$ in all the inequalities we actually get equalities.

Example: $(\mathbb{R}^2, \|\cdot\|_{\infty})$, $x = (0,1)$, $y = (0,-1)$.

$\eta_1(t) = (0, 1-2t)$ is a geodesic from x to y

$\eta_2(t) = \begin{cases} (2t, 1-2t) & t \in (0, 1/2) \\ (2(1-t), 1-2t) & t \in (1/2, 1) \end{cases}$ is a geodesic.



Theorem: Let $X = \mathbb{R}^d$, $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$. Let γ optimal coupling for $\mathcal{W}_p$.

Then, a Wasserstein geodesic between μ and ν is $\mu_t = \pi_t \# \gamma$,

where $\pi_t(x, y) = (1-t)x + ty$.

I consider the geodesic distance on the manifold

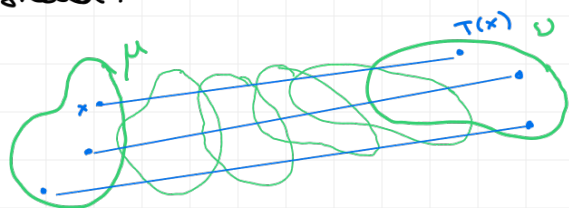
Remark: in the particular case $\gamma = (\text{Id}, T) \# \mu$, it is induced by a map:

$$\mu_t = \pi_t \# (\text{Id}, T) \# \mu = (\pi_t \circ (\text{Id}, T)) \# \mu = T_t \# \mu,$$

$$\pi_t \circ (\text{Id}, T)(x) = \pi_t(x, T(x)) = (1-t)x + tT(x) =: T_t(x).$$

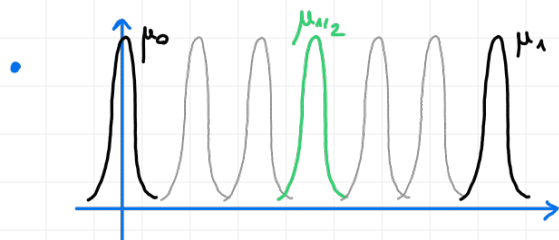
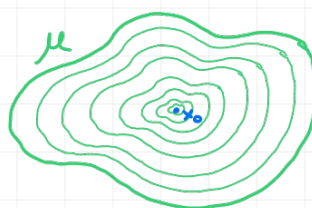
Morally, to find the W_p geodesic we should:

- take optimal map from μ to ν .
- see initial measure as collection of point-like masses.
- let each point x move along the geodesic to its final destination $T(x)$ (at constant speed).



Examples: • Geodesic from δ_x to δ_y ?

- Geodesic from δ_{x_0} to μ ?



Instead what would be the mid-point of the geodesic with resp. to $\|\cdot\|_{L^1}$?

Proof of theorem: $s < t$. Define $\gamma_{st} = (\pi_s, \pi_t) \# \gamma \in \mathcal{P}(\mu_s, \mu_t)$:

$$\pi_s \# \gamma_{st} = \pi_s \# \gamma = \mu_s.$$

$$(1-s)x + sy - (1-t)x - ty = (s-t)(x-y)$$

$$\begin{aligned} W_p(\mu_s, \mu_t) &\leq \left(\int_{x \times x} |z - z'|^p d\gamma_{st}(z, z') \right)^{1/p} = \left(\int_{x \times x} |\pi_s(x, y) - \pi_t(x, y)|^p d\gamma(x, y) \right)^{1/p} \\ &= |s-t| \left(\int_{x \times x} |x - y|^p d\gamma(x, y) \right)^{1/p} = |s-t| W_p(\mu_0, \mu_1) \quad (*) \end{aligned}$$

$$W_p(\mu_0, \mu_1) \leq W_p(\mu_0, \mu_s) + W_p(\mu_s, \mu_t) + W_p(\mu_t, \mu_1) \stackrel{(*)}{\leq} (s + (t-s) + 1-t) W_p(\mu_0, \mu_1).$$

→ All the inequalities are equalities.

Def: F is geodesically convex if $\forall x, y, \exists \eta: [0, 1] \rightarrow X$ constant speed geodesic from x to y s.t. $F \circ \eta: [0, 1] \rightarrow \mathbb{R} \cup \{+\infty\}$ is convex, i.e.,

$$F(\eta(t)) \leq (1-t)F(x) + tF(y).$$